

On a possible compensation of the QCD vacuum contribution to the Dark Energy

Roman Pasechnik

*Department of Astronomy and Theoretical Physics,
Lund University, SE-223 62 Lund, Sweden*

Vitaly Beylin

*Research Institute of Physics, Southern Federal University,
344090 Rostov-on-Don, Russian Federation*

Grigory Vereshkov

*Research Institute of Physics, Southern Federal University,
344090 Rostov-on-Don, Russian Federation and*

*Institute for Nuclear Research of Russian Academy of Sciences,
117312 Moscow, Russian Federation*

Abstract

We demonstrate one of the possible ways to compensate the large negative quantum-topological QCD contribution to the vacuum energy density of the Universe by means of a positive constant contribution from a cosmological Yang-Mills field. An important role of the exact particular solution for the Yang-Mills field corresponding to the finite-time instantons is discussed.

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I. INTRODUCTION

Current accelerated expansion of the Universe is commonly attributed to the existence of the so-called Dark Energy which is confirmed in many cosmological observations so far, e.g. in studies of the type Ia Supernovae [1], cosmic microwave background anisotropies [2], large scale structure [3] etc. The Standard Cosmological Model is based on the time-independent Dark Energy approximation called the cosmological constant, or Λ -term, approximation which agrees well with current observational data. However, the problem of theoretical interpretation and prediction of fundamental properties of the Dark Energy (or the Λ -term) remains one of the major unsolved problem of Theoretical Physics [4]. For a comprehensive overview of existing theoretical models and interpretations of the Dark Energy, see e.g. Refs. [5–8] and references therein.

One of the traditional interpretations of the Λ -term is by means of the vacuum energy density satisfying the equation of state $P_\Lambda = -\rho_\Lambda$ with vacuum pressure P_Λ and energy density ρ_Λ . However, individual vacuum condensates known from particle physics e.g. those which are responsible for the chiral and gauge symmetries breaking in the Standard Model, contribute to the vacuum energy of the Universe individually exceeding the observable value of the Λ -term density $\Lambda_{\text{exp}} = (3.0 \pm 0.7) \times 10^{-35} \text{ MeV}^4$ [2] by many orders of magnitude in absolute value [9]. This situation, which sometimes referred to as the “Vacuum Catastrophe” in the literature, requires extra hypotheses about (partial or complete) compensation of vacuum condensates of different types to the net vacuum energy density of the Universe (see e.g. Ref. [10]). A dynamical mechanism for such huge cancelations and corresponding major fine-tuning of vacuum parameters is yet not known and is a subject of ongoing intensive studies (for a review on this topic, see e.g. Ref. [5] and references therein).

Within the general problem of vacuum condensates cancelation, the QCD vacuum contribution has a special status. Various existing cancelation mechanisms refer essentially to an unknown high-scale physics beyond the Standard Model e.g. to Supersymmetry [5]. However, they cannot be applied for a compensation of the specifically non-perturbative and low-energy QCD contribution. In this paper, we focus primarily on elimination of this most “difficult” part of the vacuum energy of the Universe.

In the framework of the popular instanton liquid models [11], the topological (or instanton) modes of the QCD vacuum (which sometimes referred to as the quark-gluon condensate) are given essentially by the strong non-perturbative fluctuations of the gluon and light sea quark fields which are induced in processes of quantum tunneling of the gluon vacuum between topologically different classical states. The topological instanton-type contribution $\varepsilon_{vac(top)}$ to the energy density of the QCD vacuum ε_{vac} is one of its main characteristics [12] and can be written as follows (see also Ref. [13])

$$\begin{aligned} \varepsilon_{vac(top)} = & -\frac{9}{32} \langle 0 | : \frac{\alpha_s}{\pi} F_{\mu\nu}^a(x) F_a^{\mu\nu}(x) : | 0 \rangle + \frac{1}{4} \left[\langle 0 | : m_u \bar{u}u : | 0 \rangle + \langle 0 | : m_d \bar{d}d : | 0 \rangle \right. \\ & \left. + \langle 0 | : m_s \bar{s}s : | 0 \rangle \right] \simeq -(5 \pm 1) \times 10^9 \text{ MeV}^4, \end{aligned} \quad (1.1)$$

which is composed of gluon and light sea u, d, s quark contributions. Clearly, other contributions of a different physical nature should compensate the topological QCD contribution (1.1) to the vacuum energy of the Universe since its value by far is not compatible with the cosmological observations and data on the Λ -term value [2]. This issue triggers the search for possible cancelation mechanisms, and one such mechanism will be discussed further in this paper.

II. CLASSICAL EVOLUTION OF THE COSMOLOGICAL YANG-MILLS FIELDS

Consider one of the possible ways to eliminate the *microscopic* QCD vacuum contribution (1.1) to the vacuum energy density of the Universe introducing the hypothesis about the existence of the cosmological *macroscopic* Yang-Mills fields in early Universe.

Cosmological solutions for classical Yang-Mills fields have a long history referring back to the late seventies, when there was an active search for solutions to the Einstein-Yang-Mills field equations [14]. Later, the role of non-Abelian gauge fields in the early Universe evolution has been intensively studied in many different aspects, in particular, in the context of the Dark Energy [15] and non-Abelian fields driven inflation without a presence of a scalar field (“gauge-flation”) [16]. Practically, there are no any physical arguments which could forbid the existence of the homogeneous non-Abelian gauge field with unbroken $SU(N)$ symmetry at cosmological scales with an isotropic energy-momentum tensor [17], possibly originating from the inflationary stage of the Universe evolution [16].

Let us now assume that a chromodynamical (gluon) field with unbroken color $SU(3)_c$ symmetry exists as a real physical object filling up the early Universe, and the subject of our further discussion concerns possible physical states of this field and their real-time dynamics. For simplicity, we work in the flat Friedmann Universe with conformal metric $g_{\mu\nu} = a^2(\eta)g_{\mu\nu(M)}$, where $g_{\mu\nu(M)}$ is the Minkowski metric. The Einstein equations with energy-momentum tensor of classical Yang-Mills fields are [14]

$$\begin{aligned} \frac{1}{\varkappa} \left(R^\nu_\mu - \frac{1}{2} \delta^\nu_\mu R \right) &= \frac{1}{g_{\text{YM}}^2} \frac{1}{\sqrt{-g}} \left(-F_{\mu\lambda}^a F_a^{\nu\lambda} + \frac{1}{4} \delta^\nu_\mu F_{\sigma\lambda}^a F_a^{\sigma\lambda} \right), \quad \sqrt{-g} = a^4(\eta), \\ \left(\frac{\delta^{ab}}{\sqrt{-g}} \partial_\nu \sqrt{-g} - f^{abc} A_\nu^c \right) \frac{F_b^{\mu\nu}}{\sqrt{-g}} &= 0, \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c. \end{aligned} \quad (2.1)$$

This system is written in the most trivial form without taking into account interactions of the macroscopic Yang-Mills field with the physical vacuum (no vacuum polarisation effects are included here) and other forms of matter (i.e. $\varepsilon = 0$). Here and below, raising and lowering Lorenz indices are done by the Minkowski metric $g_{\mu\nu(M)}$ as usual.

Since initial conditions in the early Universe are quite arbitrary, it is meaningful to start with the study of spatially-homogeneous and isotropic modes of the gluon field [17]. A specific feature of such modes concerns their distinct topological structure where the isotopic and spatial indices are mixed up. In the case of Hamiltonian gauge $A_0^a = 0$ and homogeneous and isotropic 3-space we have the following simple structure of these modes:

$$A_i^a = \begin{cases} \delta_i^a A(\eta), & i, a = 1, 2, 3 \\ 0, & i = 1, 2, 3; a > 3, \end{cases} \quad (2.2)$$

with a single non-trivial time-dependent degree of freedom $A(\eta)$ to be studied in what follows. In this case, the classical Yang-Mills equations (2.1) read

$$\frac{3}{\varkappa} \frac{a'^2}{a^4} = \frac{3}{2g_{\text{YM}}^2 a^4} (A'^2 + A^4), \quad A'' + 2A^3 = 0, \quad (2.3)$$

and thus completely determine the conformal time evolution of the spatially-homogeneous and isotropic Yang-Mills field. The second equation in Eq. (2.3) can be exactly integrated, and its general solution implicitly corresponds to non-linear oscillations, i.e.

$$A'^2 + A^4 = C^4, \quad \int_{A_0}^A \frac{dA}{\sqrt{C^4 - A^4}} = \eta, \quad (2.4)$$

with C, A_0 being integration constants. Numerical solution of Eq. (2.3) for the gluon field potential with initial condition $A'(0) = 0$ and an arbitrary amplitude $A_0 = C$ to a good accuracy can be approximated by

$$A(\eta) \simeq A_0 \cos\left(\frac{6}{5} A_0 \eta\right). \quad (2.5)$$

An essentially non-linear character of oscillations of the classical YM field is thus emerged in explicit dependence of their amplitude on frequency. According to Eqs. (2.3) and (2.4), the spatially-homogeneous classical YM field in the isotropic Universe behaves as an ultra-relativistic medium with energy density $\varepsilon_{\text{YM}} \sim 1/a^4$ and equation of state $p_{\text{YM}} = \varepsilon_{\text{YM}}/3$ [17].

III. ROLE OF THE VACUUM POLARISATION

Can a classical Yang-Mills field be a component of the cosmological medium in the radiation-dominated Universe? A simple analysis have shown that the classical spatially-homogeneous Yang-Mills field cannot exist in the early Universe since the classical Yang-Mills equations (2.1) are not form-invariant and unstable with respect to radiative corrections. Such an instability emerges due to the fact that there is no any threshold for vacuum polarization of a massless non-linear gauge field, i.e. any infinitesimally small external field is capable of reconstruction of the classical Yang-Mills vacuum [18]. Due to non-linearity of initial operator Yang-Mills equations the vacuum polarization of the massless quantum gluon field by its classical component leads to a modification of classical equations. In practice, we deal with the Savvidy equations for the Savvidy vacuum fluctuations [18] and look for their spatially-homogeneous modes.

Next, let us analyze the Yang-Mills equations incorporating the vacuum polarisation effects. The Lagrangian of the gluon field taking into account the vacuum polarisation in the one-loop approximation has the following form [18]:

$$\begin{aligned} L_{\text{YM}} &= -\frac{1}{4g_{\text{YM}}^2} \frac{F_{\alpha\beta}^a F_a^{\alpha\beta}}{\sqrt{-g}} \left[1 + \frac{\beta}{2} \ln\left(\frac{J}{\Lambda_{\text{QCD}}^4}\right) \right] \\ &= -\frac{11}{128\pi^2} \frac{F_{\mu\nu}^a F_a^{\mu\nu}}{\sqrt{-g}} \ln\left(\frac{J}{\Lambda_{\text{QCD}}^4}\right), \quad J = \frac{1}{\xi^4} \frac{|F_{\alpha\beta}^a F_a^{\alpha\beta}|}{\sqrt{-g}}. \end{aligned} \quad (3.1)$$

Here, the numerical parameter ξ is not fixed and reflects an ambiguity in normalisation of the corresponding gauge/Lorentz invariant J . Such a Lagrangian leads to a modified system of equations for gravitational and Yang-Mills fields in the isotropic Universe with vacuum polarisation effects incorporated, namely,

$$\begin{aligned} \frac{1}{\varkappa} \left(R_\mu^\nu - \frac{1}{2} \delta_\mu^\nu R \right) &= T_\mu^{\nu, \text{mat}} + \bar{\Lambda} \delta_\mu^\nu + \frac{11}{32\pi^2} \frac{1}{\sqrt{-g}} \left[\left(-F_{\mu\lambda}^a F_a^{\nu\lambda} \right. \right. \\ &\quad \left. \left. + \frac{1}{4} \delta_\mu^\nu F_{\sigma\lambda}^a F_a^{\sigma\lambda} \right) \ln \frac{e |F_{\alpha\beta}^a F_a^{\alpha\beta}|}{\sqrt{-g} (\xi \Lambda_{\text{QCD}})^4} - \frac{1}{4} \delta_\mu^\nu F_{\sigma\lambda}^a F_a^{\sigma\lambda} \right], \\ \left(\frac{\delta^{ab}}{\sqrt{-g}} \partial_\nu \sqrt{-g} - f^{abc} A_\nu^c \right) &\left(\frac{F_b^{\mu\nu}}{\sqrt{-g}} \ln \frac{e |F_{\alpha\beta}^a F_a^{\alpha\beta}|}{\sqrt{-g} (\xi \Lambda_{\text{QCD}})^4} \right) = 0, \end{aligned} \quad (3.2)$$

where $e \simeq 2.71$ is the base of the natural logarithm; Λ_{QCD} is the QCD energy scale; $T_\mu^{\nu, \text{mat}} = (\varepsilon + p)u_\mu u^\nu - \delta_\mu^\nu p$ is the energy-momentum tensor of all components of the cosmological medium except for the macroscopic Yang-Mills field; $\bar{\Lambda} = \Lambda_{\text{inst}} + \Lambda_{\text{cosm}} + \dots$ is the total contribution to the vacuum energy density which consists of the *non-perturbative* spatially-inhomogeneous (topological) quantum fluctuations of the gluon and quark fields (quark-gluon condensate) of an instanton nature (1.1), $\Lambda_{\text{inst}} \equiv \varepsilon_{\text{vac}(top)} \simeq -(5 \pm 1) \times 10^9 \text{ MeV}^4$, an uncompensated contribution from the observable cosmological Λ -term, Λ_{cosm} , and dots represent all other *perturbative* vacua contributions. The Λ -term value, Λ_{cosm} , could have a different nature, other than topological non-perturbative one in QCD or perturbative ones in high-energy particle physics, so we explicitly separated it from the rest. From now on, we implicitly assume that perturbative components of the net vacuum energy density from all other microscopic vacuum condensates in particle physics are compensated elsewhere at high energy scales and do not enter the vacuum energy density of the Universe, so $\bar{\Lambda} = \Lambda_{\text{inst}} + \Lambda_{\text{cosm}}$.

The system of equations (3.2) is written in the most general form including all forms of matter, as well as the uncompensated quark-gluon condensate contribution Λ_{inst} and the observable cosmological Λ -term Λ_{cosm} . The components of the energy-momentum tensor for the homogeneous and isotropic modes specified in Eq. (2.2) have the following generic form:

$$\begin{aligned} T_0^{0, \text{tot}} &= T_0^{0, \text{mat}} + \bar{\Lambda} + \frac{33}{64\pi^2} \frac{1}{a^4} \left[(A'^2 + A^4) \ln \frac{6e|A'^2 - A^4|}{a^4(\xi\Lambda_{\text{QCD}})^4} + A'^2 - A^4 \right], \quad T_0^{\beta, \text{tot}} = T_0^{\beta, \text{mat}}, \\ T_\alpha^{\beta, \text{tot}} &= T_\alpha^{\beta, \text{mat}} + \bar{\Lambda} \delta_\alpha^\beta + \frac{11}{32\pi^2} \frac{1}{a^4} \delta_\alpha^\beta \left[-\frac{1}{2} (A'^2 + A^4) \ln \frac{6e|A'^2 - A^4|}{a^4(\xi\Lambda_{\text{QCD}})^4} + \frac{3}{2} (A'^2 - A^4) \right] \end{aligned} \quad (3.3)$$

In flat and isotropic Universe, trace of the Einstein equations and the equation of motion of the macroscopic gluon field read, respectively,

$$\frac{6}{\varkappa} \frac{a''}{a^3} = \varepsilon - 3p + 4\bar{\Lambda} + T_\mu^{\mu, \text{YM}}, \quad T_\mu^{\mu, \text{YM}} = \frac{33}{16\pi^2} \frac{1}{a^4} (A'^2 - A^4), \quad (3.4)$$

$$\frac{\partial}{\partial \eta} \left(A' \ln \frac{6e|A'^2 - A^4|}{a^4(\xi\Lambda_{\text{QCD}})^4} \right) + 2A^3 \ln \frac{6e|A'^2 - A^4|}{a^4(\xi\Lambda_{\text{QCD}})^4} = 0. \quad (3.5)$$

It is straightforward to show that the (0, 0) Einstein equation

$$\frac{3}{\varkappa} \frac{a'^2}{a^4} = \varepsilon + \bar{\Lambda} + \frac{33}{64\pi^2} \frac{1}{a^4} \left[(A'^2 + A^4) \ln \frac{6e|A'^2 - A^4|}{a^4(\xi\Lambda_{\text{QCD}})^4} + A'^2 - A^4 \right] \quad (3.6)$$

is the exact first integral of the system of equations (3.4) and (3.5), while the exact first integral of second equation (3.5) is

$$\frac{6e(A'^2 - A^4)}{a^4(\xi\Lambda_{\text{QCD}})^4} = 1. \quad (3.7)$$

The latter leads to a considerable simplification of the energy-momentum tensor, namely,

$$\begin{aligned} T_0^{0, \text{tot}} &= T_0^{0, \text{mat}} + \bar{\Lambda} + \frac{33}{64\pi^2} \frac{(\xi\Lambda_{\text{QCD}})^4}{6e}, \\ T_\alpha^{\beta, \text{tot}} &= T_\alpha^{\beta, \text{mat}} + \left(\bar{\Lambda} + \frac{33}{64\pi^2} \frac{(\xi\Lambda_{\text{QCD}})^4}{6e} \right) \delta_\alpha^\beta. \end{aligned} \quad (3.8)$$

Now we can observe an interesting possibility to eliminate the microscopic *negative QCD contribution* to the vacuum energy, Λ_{inst} , by means of the *constant positive contribution* from the spatially-homogeneous mode of macroscopic gluon field. A small non-compensated remnant – the observable Λ -term, Λ_{cosm} – can, in principle, have a different nature which will be discussed in our forthcoming publication. The corresponding condition for the $\Lambda_{\text{inst}} \simeq -265^4 \text{ MeV}^4$ compensation

$$\frac{33}{64\pi^2} \frac{(\xi \Lambda_{\text{QCD}})^4}{6e} + \Lambda_{\text{inst}} = 0, \quad \Lambda_{\text{QCD}} \simeq 280 \text{ MeV}, \quad (3.9)$$

however, is not fully automatic; it is satisfied for a certain value of the normalisation parameter only, $\xi \simeq 4$, which should be constrained in a complete theory of the QCD vacuum. Therefore, in principle, one succeeds to eliminate the huge negative contribution from spatially-inhomogeneous non-perturbative quantum fluctuations of the gluon field by means of a positive contribution from fluctuations of spatially-homogeneous macroscopic gluon field. This is achieved by fixing the remaining freedom in normalization of the Yang-Mills invariant J in the Lagrangian (3.1). As we will see below, both mutually compensating contributions to the vacuum energy density of the Universe have a common instanton nature.

IV. COSMOLOGICAL EVOLUTION OF FINITE-TIME INSTANTONS

Together with the compensation condition (3.9) and the first integrals (3.6) and (3.7), the resulting system of equations (3.4) and (3.5) is dramatically reduced to the following simple form:

$$\frac{3}{\varkappa} \frac{a'^2}{a^4} = \varepsilon + \Lambda_{\text{cosm}}, \quad (4.1)$$

$$A'^2 - A^4 = a^4 \frac{(\xi \Lambda_{\text{QCD}})^4}{6e}. \quad (4.2)$$

Notice that under the exact cancelation condition (3.9) the cosmological (macroscopic) evolution of the Friedmann Universe given by the scale factor $a = a(\eta)$ is now completely decoupled from the microscopic evolution of the gluon field $A = A(\eta)$. The physical time scale for the cosmological evolution is of the order of the Universe age $t_{\text{cosm}} \sim 1/H$ (in terms of the Hubble parameter H), while the typical time scale for the Yang-Mills field evolution is of the order of the hadronisation time $t_{\text{hadr}} \sim 1/\Lambda_{\text{QCD}}$. So at present epoch the right hand side of Eq. (4.2) can be taken to be constant in time to a good accuracy, or more precisely, given by a classical solution of the Friedmann equation (4.1). In practice, this means that the dynamical cancelation under the condition (3.9) and, hence, the decoupling of the QCD vacuum fluctuations from the hot cosmological plasma have effectively happened at the end of the hadronisation epoch in the early Universe evolution.

For convenience, let us rewrite the Yang-Mills equation (4.2) in terms of dimensionless time and gauge field as follows

$$\left(\frac{d\tilde{A}}{d\tilde{\eta}} \right)^2 - \tilde{A}^4 = 1, \quad \tilde{A} = A \frac{(6e)^{1/4}}{\xi \Lambda_{\text{QCD}}} \simeq \frac{A}{2\Lambda_{\text{QCD}}}, \quad \tilde{\eta} = \eta \frac{\xi \Lambda_{\text{QCD}}}{(6e)^{1/4}} \simeq 2\Lambda_{\text{QCD}} \eta, \quad (4.3)$$

for $\xi \simeq 4$. For simplicity, we have chosen the initial values of the Yang-Mills field and the scale factor as follows:

$$\tilde{A}(\tilde{\eta}_0 = 0) \equiv \tilde{A}_0 = 0, \quad a(\tilde{\eta}_0 = 0) \equiv a_0 = 1. \quad (4.4)$$

We can see now that, indeed, the time scale of the Yang-Mills field fluctuations is essentially microscopic and corresponds to the Λ_{QCD} energy scale. The equation (4.3) can then be easily integrated, and its general solution can be written in the following implicit form:

$$\int_{\tilde{A}_0}^{\tilde{A}} \frac{d\tilde{A}}{\sqrt{1 + \tilde{A}^4}} = \tilde{\eta}, \quad (4.5)$$

where \tilde{A}_0 is an integration constant. Notably, the analytical solution (4.5) taking into account the QCD vacuum polarisation, in fact, differs from the classical Yang-Mills solution (2.4) by sign in front of \tilde{A}^4 under the square root only, having though a significant effect on its time dependence. Moreover, since the solution (4.5) was obtained under the exact cancelation condition (3.9), it corresponds to the minimal energy of the QCD system in the ground state of the Universe and, hence, is physically preferable.

For the initial conditions given by Eq. (4.4) (independently on a particular \dot{a}_0 value), the solution for $\tilde{A}(\tilde{\eta})$ (4.5) obeys the following properties:

- Symmetry: $\tilde{A}(-\tilde{\eta}) = -\tilde{A}(\tilde{\eta})$.
- Periodicity: $\tilde{A}(\tilde{\eta} \pm T) = \tilde{A}(\tilde{\eta})$.
- Continuous intervals and singularities: $\tilde{A}(\tilde{\eta} \rightarrow \pm T/4) = \pm\infty$.

Most importantly, it is continuous only at a finite microscopically-small time interval $T \sim 1/\Lambda_{\text{QCD}}$ and corresponds to spatially-homogeneous pulses of the gluon field potential with a constant energy-density. We notice also that once the compensation condition (3.9) has been satisfied at a particular moment in time, it holds true for any later times, and the YM fields and large negative Λ_{inst} disappear from the resulting equations and do not participate in the Universe evolution any longer. We, therefore, arrive at quasistationary regime when the Universe evolution is completely determined by usual matter and uncompensated cosmological Λ -term only.

What is the physical interpretation of the result (4.5)? Obviously, such a solution with regular singularities does not have a quasiclassical interpretation (except for a vicinity of the midpoints of the continuous intervals where the gluon field potential is small and slowly changing). In practice, we deal with a sequence of quantum fluctuations of the YM field in time, or, in fact, with *the finite-time instantons*. The creation and annihilation of such finite-time instantons should have essentially quantum nature like the QCD instantons. In order to regularize singularities in the quasiclassical solution (4.5) one needs to turn into a complete quantum theory taking into account quantum corrections due to e.g. fermion-antifermion pair creation and annihilation processes in the early Universe.

V. CONCLUSION

Based on the quasiclassical result (4.5) only, one can naively conjecture that *the cosmological evolution of the YM field emerges a sequence of quantum tunneling transitions through the time barriers* represented by the regular singularities in the solution (4.5). Thus, we have observed that the positive constant energy density of spatially-homogeneous finite-time instantons in the early Universe can be canceled with the negative constant QCD contribution

from spatially-inhomogeneous gluon field fluctuations induced by a similar quantum tunneling of the gluon field, but through spatial (not time!) topological barriers between different classical vacua. This exhibits a remarkable similarity and interplay between instantons of different types in the early Universe evolution.

From the point of view of quantum tunneling, the chain of quantum fluctuations in a certain approximation can be considered as non-linear oscillations. In order to regularize the infinite end-points of the continuous time intervals within the quasiclassical approach, one could therefore consider a continuous smearing of the resulting fluctuations by means of a non-linear continuous parameterization such as

$$\tilde{A}_{appr}(\tilde{\eta}) = \frac{1}{a \sin(\omega\tilde{\eta}) + b \frac{\cos^2(\omega\tilde{\eta})}{\sin(\omega\tilde{\eta})}} \quad (5.1)$$

with adjustable parameters a , b and ω . This approximation has been qualitatively compared to the exact quasiclassical solution (4.5) and approaches it in the limit of small $a \rightarrow 0$, while the classical non-linear solution (2.4) is reached in the limit $a \rightarrow b$ (up to an arbitrary initial phase) with ω being dependent on the initial amplitude. Thus, the parameterization (5.1) represents a simple continuous interpolation between the classical and quasiclassical solutions, and can be used in practical calculations in the quasiclassical limit of the theory. A detailed quantum analysis of the finite-time instantons is certainly a necessary step forward and is planned for a forthcoming study.

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